SCI. REP. KYOTO PREF. UNIV. (NAT. SCI. & LIV. SCI.), NO. 33, Ser. A, p. 1~8 (NOV. 1982)

Discrete Potential and its Properties I

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(Received July 12, 1982)

We expand properties about the discrete potential theory. The lattice functions, take the values only at the lattice points of the N-dimensional Euclidean space R^N $(N \geqq 2)$, defined on the lattice (set of lattice points) X have analogous behavior for continuous functions under certain conditions, still more are closely connected with harmonic- and superharmonic-functions, Green's functions and potentials etc. In this paper we investigate the representation of a solution of the Dirichlet problem and properties of Green's functions.

Introduction.

The object of this paper is to explore certain properties of the lattice functions, and to investigate under what conditions well-known properties of the continuous potential theory are extended to the discrete potential theory. For a discussion of the significance of various properties of the continuous potential theory see Helms [9]. We shall find that many but not all the classical theorems remain valid for the discrete potential theory.

We define the corresponding operator L to the Laplacian differential operator Δ. This operator L operates on lattice functions defined on the lattice (set of lattice points) of R^N. In example, a lattice function $u(p)$ at lattice points $p=p(x_1, x_2)$ in R^2 which are restricted to rational integers is discrete harmonic if it satisfies the difference equation

$$Lu(p) := \{u(p_1)+u(p_2)+u(p_3)+u(p_4)-4u(p)\} = 0,$$

where lattice points p_i, $i=1, 2, 3, 4$, satisfy the condition:

$$\text{Euclidean metric dist}(p, p_i)=1.$$

Such operators employ the very important role in physical problems and in probability problems (see Feller [7]).

In this paper we consider the minimum principle and obtain the representation of the solution of a Dirichlet problem in §2. In §3, for the finite connected lattice X with the non-void set X^0 of all interior points of X the Green's function is defined. We study some interesting questions concerning the Green's function and the Green's potential.

1. Difinitions and Basic Concepts.

Let X stand for the set of lattice points in the N-dimensional Euclidean space R^N $(N \geqq 2)$. The set X, which consists of finite number points on the lattice, is called the discrete compact space. Let us denote by p, q, $\cdots$ the points of the coordinate $(x_1, x_2, \cdots, x_N)$ in R^N. The lattice point of this space has the coordinate $(l_1 r, l_2 r, \cdots, l_N r)$, where $l_1, l_2, \cdots, l_{N-1}$ and l_N take values $0, \pm 1, \pm 2, \cdots$ and r is the given positive constant. Two points p and q in X will be

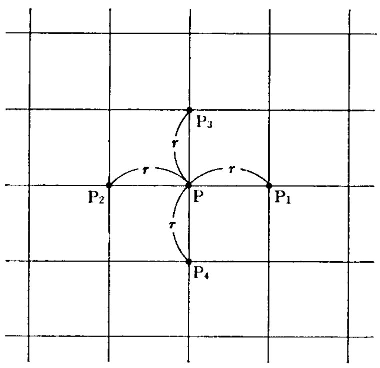

Fig. 1.

called *r-neighbouring points* (merely *neighbouring points*) if

$$\text{dist}\,(p, q)=r,$$

where "dist" denotes the Euclidean metric in R^N. Then the set, denote by U_p, of r-neighbouring points with respect to a point p in R^N consists of finite number points, especially 2N points. In example, let X be the finite compact set in R^2 (2-dimensional Euclidean space). Then any lattice point $p=p(x_1, x_2)$ has four neighbouring points $p_1(x_1+r, x_2)$, $p_2(x_1-r, x_2)$, $p_3(x_1, x_2+r)$ and $p_4(x_1, x_2-r)$, and U_p is equal to the set $\{p_1, p_2, p_3, p_4\}$ (see Fig. 1).

Let X be a finite and connected lattice of R^N. This is a mean that a set M of lattice points is called "connected" if any two points of M can be connected by a chain of neighbouring points which belong to the set M, and also called if p_1 is a neighbour of p_2, p_2 is a neighbour of p_3, etc.$\cdots$ for $p_1, p_2, p_3, \cdots$ belonged to M.

A lattice point p in X is called the interior point of X if the set U_p of neighbouring points of p is entirely contained in X. Let us denote the set of interior points of X by X^0, and $X\cap CX^0$ by X^*, which is the set of so-called boundary points of X. Generally the considering lattice X consists of such two types of points.

It should be noted that a domain X^0 is not uniquely determined as a set of points, unless some rule is given to distinguish the interior points of the set X (see Inoue [10] and Heilbronn [8]). A connected lattice X will be called finite if it contains only a finite number of points, otherwise it will be infinite.

A lattice function $u(p)$ is a numerical function defined only at the lattice points of R^N. The operator L is defined on the family of lattice functions by

$$(1)\qquad Lu(p):=\frac{1}{r^2}\{u(p_1)+u(p_2)+u(p_3)+\cdots+u(p_{2N})-2Nu(p)\}$$

at the lattice points in R^N. It is clear that this operator is linear. The operator L may be termed the *Laplacian*. For a lattice function $\varphi(p)$, let us consider the so-called *Poisson's equation*:

$$(2)\qquad Lu=-\varphi.$$

If φ vanishes, this becomes *Laplace's equation*:

$$(3)\qquad Lu(p)=0.$$

A lattice function u is said to be a *discrete harmonic function* in the lattice X if it is defined for all points of X and if the equation (3) is held for all interior points p of X.

Now let X be assumed to be a connected lattice with more interior points than one. We define the following discrete probability measure μ_p for each point p in X^0, which is carried on the set U_p of neighbouring points of p,

(4) $$\mu_p := \frac{1}{k} \sum_{i=1}^{k} \varepsilon_{p_i} \qquad \text{for } p_i \in U_p,$$

where "k" denotes the number of all points belonged to U_p, here $k=2N$ in R^N.

Thus a lattice function u defined on X is discrete harmonic at an interior point p if and only if its value at p is equal to the integral mean with respect to the discrete probability measure μ_p, (4), carried on the neighbouring point set U_p of p,

(5) $$u(p) = \int u \, d\mu_p \qquad \text{for } p \in X^0.$$

Moreover, if the lattice function s defined on X satisfies the condition:

(6) $$s(p) \geqq \int s \, d\mu_p \qquad \text{for } p \in X^0$$

with respect to a discrete probability measure μ_p carried on the neighbouring set U_p, s is said to be a *discrete superharmonic function.* This definition is equivalent to the property:

(7) $$Ls(p) \leqq 0 \qquad \text{for } p \in X^0$$

with respect to the operator L.

2. Dirichlet problem and Poisson formula.

Let X be a finite compact lattice in R^N with the non-void interior point set X^0. We have the minimum principle for a discrete superharmonic function on X.

Theorem 1. *Let $s(p)$ be a lattice function defined on X which is discrete superharmonic on a finite domain X^0 such that $s > -\infty$ on X. Then s is either the constant on X or it attains the minimum value of s on the boundary points of X.*

Proof. Let us set the constant

$$m := \inf_{p \in X} s(p).$$

Now we assume that there exists a point p_0 belonged to X^0 such that $s(p_0)=m$. Then from the definition of a discrete superharmonic function we may have the neighbouring set U_{p_0} of p_0 in X^0 such that

$$s(p_0) \geqq \int s \, d\mu_{p_0},$$

where the measure μ_{p_0} denotes the discrete probability measure carried on the U_{p_0}, that is

$$\mu_{p_0} = \frac{1}{2N} \sum_{i=1}^{2N} \varepsilon_{p_i}$$

for $p_i \in U_{p_0}$, $i=1, 2, \cdots, 2N$.

Thus

$$m = s(p_0) \geqq \int s \, d\mu_{p_0} = \frac{1}{2N} \{s(p_1) + s(p_2) + \cdots + s(p_{2N})\}$$

and for each $p_i \in U_{p_0}$, $i=1, 2, \cdots, 2N$,

$$s(p_i) \geqq m.$$

Therefore since

$$s(p_1)+s(p_2)+\cdots+s(p_{2N})=2Nm,$$

we get

$$s(p_0)=s(p_1)=s(p_2)=\cdots=s(p_{2N})=m.$$

By the induction the function s is constant at all points of X as the set X^0 consists of the finite numbers of points. Hence the function s is the constant for all $p\in X$, which completes the proof.

Theorem 2. *Let f be a given real-valued lattice function on the boundary X^* of a finite compact set X. Then there exists one and only one discrete harmonic function $u(p)$ which takes the values $f(p)$ on the boundary X^* of X.*

If $h(p)$ is a real-valued lattice function defined on X which also takes the values $f(p)$ on the boundary X^* of X, then

$$\sum_{p,q\in X^0}|u(p)-u(q)|^2=\sum_{p,q\in X^0}|h(p)-h(q)|^2$$

and the sign of equality holds only if $u(p)=h(p)$ for all $p\in X$.

See Heilbronn [8] for the proof.

Let us define the following function: K(q, p) is defined as a lattice function on the product space $X^*\times X$ with the properties;

(i) for each $q\in X^*$
K(q, p) is discrete harmonic in X^0,

(ii) for each $q\in X^*$
K(q, p) vanishes on all boundary points of X^* except at the point $p=q$

and

(iii) $K(p, p)=1$.

The function K(q, p) is non-negative on $X^*\times X$.

Lemma 3. *Let X be the finite connected lattice with some interior points of X. For a lattice function f defined on the boundary X^* of X such that*

$$f=0 \qquad \text{on } X^*\setminus\{q\}$$

and

$$f=1 \qquad \text{at } q,$$

there is a unique function K(p, q) with the properties (i), (ii) and (iii) on X.

Moreover if $u(p)$ is a discrete harmonic function on X, and if p is any point of X, the function u has the representation

$$u(p)=\sum_{q\in X^*}K(q,p)\,u(q).$$

The proof of this lemma is clear since the Dirichlet problem has a unique solution by the proceeding theorem.

Thus we get the representation of the *Poisson-formula type* for the solution of the Dirichlet problem.

Theorem 4. *Let X be a finite connected lattice with, non-void, X^0, and f be a real-valued lattice function on the boundary X^* of X. Then there exists only one discrete harmonic function u on X, this is*

$$Lu(p)=0 \qquad \text{for all } p\in X^0,$$

such that

$$(8) \qquad u(p) = \sum_{q \in X^*} K(q, p) f(q),$$

where K(q, p) is *defined on* $X^* \times X$, *which has properties* (i), (ii) *and* (iii).

3. Green's function and Green's potential.

Let X be a finite connected lattice in R^N with the non-void set X^0 consisted of all interior points of X.

The lattice function G(q, p) is defined on $X^0 \times X$, if exists, by for each $q \in X^0$, a lattice function G(q, p) on X satisfies followings:

(i) $G(q, p) = 0$ at $p \in X^*$,

(ii) $\mathcal{L}G(q, p) = 0$ at $p \in X^0$ such that $p \neq q$,

(iii) $\mathcal{L}G(q, p) + 1/r^2 = 0$ at $p \in X^0$ such that $p = q$,

and

(iv) $G(q, p) \geqq 0$.

This function G(q, p) is called a *(discrete) Green's function* with the pole q on $X^0 \times X$. It is clear that the Green's function G(q, p) is discrete superharmonic in X^0 for each $q \in X^0$. By theorem 2, if X has more interior points than one, the Green's funtion exists uniquely for X.

Theorem 5. *The Green's function* G(q, p) *defined for the finite connected lattice* X *is unique if it exists.*

Proof. Let $G_1(q, p)$ be another Green's function for X. For each point $q \in X^0$, $G_1(q, p)$ is a discrete harmonic function in X^0. Therefore for each $q \in X^0$, $G(q, p) - G_1(q, p)$ is a discrete harmonic function in X^0. By theorem 1, we have $G(q, p) = G_1(q, p)$. This completes the proof of the theorem.

Theorem 6. *If* X_1 *and* X_2, $X_1 \subset X_2$, *are finite connected lattices of* R^N *having Green's functions* G_{X_1} *and* G_{X_2}, *respectively, then*

$$G_{X_1}(q, p) \leqq G_{X_2}(q, p) \qquad \text{on } X_1{}^0 \times X_1.$$

Proof. For each $q \in X_1{}^0$,

Case 1, at $p \in X_1{}^* \cap X_2{}^*$

$$G_{X_1}(q, p) = 0 \quad \text{and} \quad G_{X_2}(q, p) = 0,$$

Case 2, at $p \in X_1{}^* \cap X_2{}^0$,

$$G_{X_1}(q, p) = 0 \quad \text{and} \quad \mathcal{L}G_{X_2}(q, p) = 0$$

and Case 3 at $p \in X_1{}^0$ and $p \neq q$

$$\mathcal{L}G_{X_1}(q, p) = 0 \quad \text{and} \quad \mathcal{L}G_{X_2}(q, p) = 0$$

and at $p \in X_1{}^0$ and $p = q$

$$\mathcal{L}G_{X_1}(q, p) = -\frac{1}{r^2} \quad \text{and} \quad \mathcal{L}G_{X_2}(q, p) = -\frac{1}{r^2}.$$

Thus for each $q \in X^0$

$$\mathcal{L}(G_{X_2}(q, p) - G_{X_1}(q, p)) = 0 \qquad \text{in } X_1{}^0,$$

and by theorem 1, Case 1 and Case 2

$$G_{X_2}(q, p) \geqq G_{X_1}(q, p) \qquad \text{on } X_1{}^0 \times X_1.$$

This completes the proof of the theorem.

We consider the *discrete measure* on the finite connected lattice E which consists of n numbers of points. This is, let μ be a measure defined on E, which carried on n points in E such that

$$\mu=\sum_{i=1}^{n} m_i \varepsilon_{p_i}, \tag{9}$$

where m_i is the maß of μ at p_i for $i=1, 2, \cdots, n$. The set of the points p_i with $m_i \neq 0$ is the support, denote by $S\mu$, of μ. If the support $S\mu$ of μ is only one point p and if the total maß of μ is 1, the measure μ is a Dirac measure ε_p.

Definition. Let X be a finite connected lattice with the non-void set X^0 which consists of n interior points. If μ is a discrete measure such that satisfies the equation (9) and is carried by the interior point set X^0, then

$$G\mu(p)=\sum_{j=1}^{n} G(q_j, p)m_j, \tag{10}$$

if defined everywhere on X^0, is called the (*discrete*) *Green's potential* of μ. Especially

$$G\varepsilon_q(p)=G(q, p),$$

which denotes the discrete potential of a point charge concentrated at the point q.

Lemma 7. *If μ is a discrete measure on the finite connected lattice* X *having a Green's function* G, *then* $G\mu$ *is discrete superharmonic on* X^0.

We can extend the some properties, whose are well-known subjects for the continuous potential, to the discrete potential.

Theorem 8. *If μ and ν are discrete measures on the finite connected lattice* X, *supported by* X^0, *for which Green's potentials* $G\mu$ *and* $G\nu$ *are defined, and if* $G\mu(p)=G\nu(p)$ *on* X^0, *then* $\mu=\nu$ *on* X.

Proof. Let us set discrete measures μ and ν such that

$$\mu=\sum_{j=1}^{n} m_{1j}\varepsilon_{q_j} \qquad \text{and} \qquad \nu=\sum_{i=1}^{n} m_{2i}\varepsilon_{q_i},$$

respectively, where X^0 consists of n interior points. Then two potentials are given in following:

$$G\mu(p)=\sum_{q_j\in X^0} G(q_j, p)m_{1j}$$

and

$$G\nu(p)=\sum_{q_i\in X^0} G(q_i, p)m_{2i}.$$

From the hypothesis of the theorem

$$\sum_{q_j\in X^0} G(q_j, p)m_{1j} = \sum_{q_i\in X^0} G(q_i, p)m_{2i},$$

this is,

$$G(q_1, p)m_{11}+G(q_2, p)m_{12}+\cdots+G(q_n, p)m_{1n}$$
$$= G(q_1, p)m_{21}+G(q_2, p)m_{22}+\cdots+G(q_n, p)m_{2n}.$$

Therefore

$$G(q_1, p)(m_{11}-m_{21})+G(q_2, p)(m_{12}-m_{22})+\cdots$$
$$\cdots+G(q_n, p)(m_{1n}-m_{2n})=0.$$

If $p=q_1$,

$$\mathcal{L}G(q_1, p)|_{p=q_1}=-\frac{1}{r^2} \qquad \text{and} \qquad \mathcal{L}G(q_l, p)|_{p=q_1}=0 \quad \text{for } l=2, \cdots, n.$$

Thus we get $m_{11}=m_{21}$, and likewise if $p=q_l$ for $l=2, 3, \cdots, n$, $m_{1l}=m_{2l}$ for $l=2, \cdots, n$, respectively. We have $\mu=\nu$,which completes the proof.

Lemma 9. *If μ is a discrete measure on the finite connected lattice X which carried by X^0, and if X have a Green's function G, such that $\mu(X)<+\infty$ and $\mu(Y)=0$ for some sub-lattice Y of X, then the Green's potential $G\mu(p)$ is discrete superharmonic on the component of X^0 containing Y.*

Lemma 10. *If, under conditions of lemma 9, the Green's potential $G\mu(p)$ of the discrete measure μ on X carried by X^0 is discrete harmonic in X^0, then μ is the zero measure.*

Proof. Since the $G\mu(p)$ is discrete harmonic in X^0, it is

$$\mathcal{L}G\mu(p)=0 \qquad \text{at } p\in X^0.$$

From the definition of the Green's potential, for the discrete measure

$$\mu=\sum_{i=1}^{n} m_i\, \varepsilon_{p_i}, \qquad \text{for } p_i\in X^0,$$

$$\mathcal{L}G\mu(p)=\mathcal{L}\Big(\sum_{i=1}^{n} G(q_i, p)m_i\Big) \tag{11}$$
$$=\sum_{i=1}^{n} m_i\cdot \mathcal{L}G(q_i, p).$$

Then if $p=q_i$ for $i=1, 2, \cdots, n$, we get $m_i=0$ for $i=1, 2, \cdots, n$. Thus $m_1=m_2=\cdots=m_n=0$, this is, μ is the zero measure. This completes the proof of the theorem.

We obtain the following theorem from above lemmas.

Theorem 11. *If $G\mu(p)$ is the discrete potential of a measure μ defined on X which is carried by X^0 having a Green's function G, then $G\mu$ is discrete harmonic on any sub-lattice of μ-measure zero.*

Proof. Now let Y be the sub-lattice of X, which is μ-measure zero. The discrete measure μ defined on X with the support $S\mu$ on X^0 is prescribed in the following:

$$\mu=\sum_{q_i\in X^0} m_i\, \varepsilon_{q_i}.$$

Therefore the Green's potential is

$$G\mu(p)=\sum_{q_i\in X^0} G(q_i, p)m_i$$
$$=\sum_{q_i\in X^0\setminus Y} G(q_i, p)m_i.$$

We have

$$\mathcal{L}G\mu(p)=\mathcal{L}\Big(\sum_{q_i\in X^0\setminus Y} G(q_i, p)m_i\Big) \tag{12}$$
$$=\sum_{q_i\in X^0\setminus Y} m_i\cdot \mathcal{L}G(q_i, p).$$

Since for $p\in Y$

$$\mathcal{L}G(q_i, p)=0 \qquad \text{for each } q_i \in X^0 \backslash Y,$$

by (12)

$$\mathcal{L}G\mu(p)=0 \qquad \text{for all } p \in Y. \tag{13}$$

We get the discrete harmonicity of $G\mu$ on Y. This completes the prroof.

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SCI. REP. KYOTO PREF. UNIV. (NAT. SCI. & LIV. SCI.), No. 33, Ser. A, p. 1∼8 (NOV. 1982)