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On Properties of Integral Kernel Functions in Axiomatic Potential Theory

by

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On the Bauer harmonic space with certain conditions in the axiomatic potential theory, we construct an integral kernel $k(x, \theta)$ and investigate properties of $k(x, \theta)$. Furthermore we show an example of $k(x, \theta)$ on one-dimentional Euclidean space R^1 in the last paragraph.

1. Introduction and preliminaries.

Recently in the abstract potential theory there are some results about integral kernel functions which give Cauchy-type integral representations of solutions of the Dirichlet problem (see Bear-Gleason [6]). In our previous paper [12] we tried to prove the existings of such kernels, which was reduced from reproducing kernel of a certain functional family on a harmonic space $(X, \mathfrak{H})$ with respect to Bauer's axioms [5]. The object of the present article is to investigate properties of integral kernel functions which are reduced in the paper [12], and an example in the Bauer harmonic space under some conditions. The results of this paper are based on a study of a kernel function with existings of a certain measure.

Let X be a connected, locally compact, non-compact Hausdorff space with countable basis, and $\mathfrak{H}(X)$ be a family of real-valued continuous functions (so-called harmonic functions) with open domains in X such that the class of harmonic functions on an open set forms a real linear space. The pair $(X, \mathfrak{H}(X))$ will be a harmonic space which satisfies the axioms I, II, III and IV of Bauer [5], and supposes the following one more axiom: The constant 1 is a superharmonic function. For any relatively compact open subset U of X we denote by $\mathfrak{H}(U)$ the set of all real continuous functions f on U, which are harmonic on U. Let $\mu_x{}^U$ be a harmonic measure with respect to a relatively compact open set U and a point x of U, that is, a balayaged measure on the complement CU of U of a Dirac measure at x. Let ν be a positive measure on a dense subset U' of U whose support $s\nu$ belongs to the closure $\bar{U}$ of U. In fact, as X is a locally compact open set U with a countable base, there is surely such a measure. Then we define a following measure on the boundary ∂U of U: $\sigma(e)=\int_U \mu_x{}^U(e)d\nu(x)$, where e is an arbitrary Borel subset of ∂U, whose existing is reduced by the constant 1 being superharmonic. Let us denote by $L^2(\sigma)$ the family of all real-valued σ-measurable functions which are defined on ∂U and $\int_{\partial U} f^2 d\sigma$ are finite. We define now following functionals on $L^2(\sigma)$: a bilinear functional

$$(f, g)_\sigma=\int_{\partial U} fgd\sigma \quad \text{for any } f, g \in L^2(\sigma)$$

and a non-negative functional

$$||f||_\sigma=\left(\int_{\partial U} f^2 d\sigma\right)^{\frac{1}{2}} \quad \text{for any } f\in L^2(\sigma).$$

Then $(f, g)_\sigma$ satisfies conditions of a scalar product and $||f||_\sigma$ is a norm under the condition that f is equal to g (denote by $f=g$) if and only if $||f-g||_\sigma=0$. It is well known that $L^2(\sigma)$ has the structure of a Hilbert space relative to the scalar product $(f, g)_\sigma$ and the norm $||f||_\sigma$.

2. Reproducing kernel and integral kernel.

For a relatively compact open set U of X, let us denote by $L^2(\mu_x^U)$ a family of real-valued functions defined on ∂U being μ_x^U-quadratic integrable. Then we have the following.

Theorem 1. *Let U be a relatively compact open set of X and σ be a positive measure on the boundary ∂U of U mentioned in the introduction and μ_x^U be a harmonic measure with respect to U and a point x of U. Then it holds*

$$L^2(\sigma)\subset \bigcap_{x\in U} L^2(\mu_x^U).$$

Proof. For any function $f\in L^2(\sigma)$, we have

$$\int_{\partial U}|f(\theta)|^2 d\sigma(\theta)=\int_{\partial U}\int_U |f(\theta)|^2 d\mu_x^U(\theta)d\nu(x)<+\infty,$$

and thus, in the dense subset U_0 of U,

$$\int_{\partial U}|f(\theta)|^2\mu_x^U(\theta)<+\infty.$$

Therefore, according to Bauer [5] Satz 1.1.8, we obtain

$$\int_{\partial U}|f(\theta)|^2 d\mu_x^U(\theta)<+\infty$$

for all $x\in U$. This completes the proof.

Next we define following functional spaces.

Definition. Let σ be a positive measure mentioned in the first paragraph then

$$R^2(U):=\left\{H_f \mid H_f(x)=\int f d\mu_x^U \text{ for all } f\in L^2(\sigma)\right\};$$

$$R^1(U):=\left\{H_g \mid H_g(x)=\int g d\mu_x^U \text{ for all } g\in L^1(\sigma)\right\},$$

where $L^p(\sigma)$, $(p=1, 2)$, denotes the family of all real-valued σ-measurable functions f on ∂U with the relation: $\int_{\partial U}|f|^p d\mu_x^U<+\infty$.

Then let us recall that $R^2(U)$ and $R^1(U)$ are subspaces of $\mathfrak{H}(U)$, and that $L^2(\sigma)$ and $R^2(U)$ are isomorphism (see Ôgawa and Murazawa [12]).

On $R^2(U)$ we define a scalar product (H_f, H_g) and a norm $||H_f||$ as follows:

$$(H_f, H_g)=(f, g)_\sigma \qquad \text{for } H_f, H_g\in R^2(U);$$
$$||H_f||=||f||_\sigma \qquad \text{for } H_f\in R^2(U).$$

Then $R^2(U)$ is a Hilbert space with respect to the scalar product (H_f, H_g) and the norm $||H_f||$.

We can obtain the following analogous thorem, concerning $R^2(U)$, to Bauer [5] Satz 1.4.4.

Theorem 2. *Let U be a relatively compact open subset of X, ν and σ be positive measures mentioned in the paragraph 1 and F be any compact subset in U. Then there exists a non-negative constant γ depending on F and σ such that*

$$\sup |u(F)|\leqq\gamma||u|| \qquad \textit{for all } u\in R^2(U).$$

Proof. See Ôgawa and Murazawa [12] Theorem 2.2.

Then we have the following theorem.

Theorem 3. *The Hilbert space $R^2(U)$ constructed above have a reproducing kernel $K(x,y)$:*

$$u(y)=(u(x),\ K(x,y)) \qquad \text{for all } u\in R^2(U).$$

Proof. By Theorem 2, we get that there exists a constant γ depending on the compact set F and σ such that for all points $y\in F$

$$|u(y)|\leqq \sup|u(F)|\leqq\gamma\|u\| \quad \text{for all } u\in R^2(U).$$

Thus the Aronszajn's condition [1] for existing of a reproducing kernel is satisfied. The proof completes.

Furthermore, from above Theorem 3 and also a fact that $R^2(U)$ and $L^2(\sigma)$ are isomorphism, we obtain immediately the following theorem.

Theorem 4. *Let U be a relatively compact open set of X, and σ be the positive measure defined in the introduction. Let $K(x,y)$ be a reproducing kernel of $R^2(U)$. Then there exists a function $k(x,\theta)$ on $U\times\partial U$ such that*

$$K(x,y)=\int_{\partial U}k(x,\theta)d\mu_y{}^U(\theta),$$

which satisfies followings:

a) *for every $x\in U$, $k(x,\theta)$ belongs to $L^2(\sigma)$;*

b) *for every $x\in U$, $k(x,\theta)$ is non-negative almost everywhere (σ) with a relation:*

$$\mu_x{}^U(e)=\int_e k(x,\theta)d\sigma(\theta)$$

for any Borel subset e of ∂U.

3. Properties of the integral kernel.

We shall now define the sweeping system on X in the method of Constantinescu and Cornea [7]. Let a pair (X, $\mathfrak{H}$) has the harmonic structure mentioned in the first paragraph. Let U be an open set of X; a family $\mathfrak{M}:=(\mu_x{}^U)_{x\in U}$ of measures on ∂U will be called a sweeping on U. The sweeping $\mathfrak{M}$ is called an $\mathfrak{H}$-sweeping if:

a) U is relatively compact;

b) for any $f\in C(\partial U)$ the function $\mu_x{}^U(f)$, for any $\mu_x{}^U\in\mathfrak{M}$, is an $\mathfrak{H}$-function;

c) for any $\mathfrak{H}$-function h defined on an open neighbourhood of $\bar{U}$ we have $\mu_x{}^U(h)=h(x)$ on U for any $\mu_x{}^U\in\mathfrak{M}$.

A sweeping system on X is a family $\Omega:=((\mu_x{}^{U_i})_{x\in U_i})_{i\in I}$ such that $\{U_i|i\in I\}$ is a base for X of relatively compact open sets and that for any $i\in I$ $\mathfrak{M}_i:=(\mu_x{}^{U_i})_{x\in U_i}$ is a sweeping on U_i. The sweeping system Ω is called an $\mathfrak{H}$-sweeping system if for any $i\in I$ $\mathfrak{M}_i$ is an $\mathfrak{H}$-sweeping.

Throughout this section, assume that there exists an $\mathfrak{H}$-sweeping system $\Omega:=((\mu_x{}^{U_i})_{x\in U_i})_{i\in I}$ on X such that for any $x\in X$ there exists $I_x\subset I$ with the following properties: a) $\{U_i|i\in I_x\}$ is a fundamental system of neighbourhoods of x; b) for any $i\in I_x$, $\partial\bar{W}_i$ is contained in the carries of $\mu_x{}^{U_i}$, where W_i denotes the component of U_i containing x. Then, if $\mathfrak{H}$ satisfies Bauer's convergence axiom [5], $\mathfrak{H}$ possesses Brelot's convergence axiom [8] (see Constantinescu and Cornea [7], and Bauer [5]). Moreover it is well known that : if there exists an $\mathfrak{H}$-sweeping system on X, then $\mathfrak{H}(X)$ is complete with respect to the topology of compact convergence.

In the sense of the Bauer harmonic space assuming that there exists an $\mathfrak{H}$-sweeping system, let us prove first the following lemma, which is proved essentially according to R.-M. Hervé [9].

Lemma 5. *Let U be a relatively compact open set of X and $\{\mu_x^U\}$ be a family of harmonic measures defined on ∂U with respect to U and points x of U, such that there exists a function $k(x, \eta)$ of $L^2(\sigma)$ for each $x \in U$ with the following relation: $d\mu_x^U(\eta)=k(x, \eta)d\sigma(\eta)$, where σ is the non-negative measure with supporting on ∂U which is defined in the first paragraph. Then, for any point $\theta \in \partial U-E$ such that $\sigma(E)=0$, there exists a decreasing sequence $\{F_n^\theta\}_n$ of compact subsets of ∂U which coverges to $\{\theta\}$, such that has following properties: $\sigma(F_n^\theta)>0$ for every n and, for every point $x \in U$ there exists $\lim_{n\to\infty} \mu_x^U(F_n^\theta)/\sigma(F_n^\theta)$, which defines the density function of μ_x^U with respect to σ, that is, $k(x, \theta)=\lim_{n\to\infty} \mu_x^U(F_n^\theta)/\sigma(F_n^\theta)$.*

Proof. Since $k_x(\eta):=k(x, \eta)$ is a density function of μ_x^U with respect to σ, which is σ-measurable, we have, according to the Lusin's theorem, that for all natural numbers n there exist compact sets $K_n' \subset \partial U$ such that

$$\sigma(\partial U-K_n') \leqq 1/n$$

and a restriction of $k_x(\eta)$ on K_n' is defined and continuous. We can now take the above sequence as a monotonically increasing sequence, and thus obtain that a set $E=\partial U-\bigcup_n K_n$ is σ-negligeble if the support of the restriction of σ on K_n' is denoted by K_n. Hence, being $\theta \notin E$, there exists at least one natural number n_0 such that $\theta \in K_{n_0}$. Then, for a decreasing sequence $\{\alpha_n^\theta\}$ of compact sets which is a fundamental system of neighbourhoods of θ, we may define the set $\alpha_n^\theta \cap K_{n_0}$ and denote it by F_n^θ, which is a decreasing sequence of compact sets of converging to $\{\theta\}$. The every set F_n^θ is of the positive measure with respect to a restriction of σ on K_{n_0}', a fortiori, of σ-measure>0.

Finely we have

$$\frac{\mu_x^U(F_n^\theta)}{\alpha(F_n^\theta)}=\frac{\int_{F_n^\theta} k(x, \eta)\, d\sigma(\eta)}{\int_{F_n^\theta} d\sigma(\eta)} \longrightarrow k(x, \theta)$$

as n increasing to ∞, because of $k(x, \eta)$ being continuous for every $\eta \in F_n^\theta$. This completes the proof of this lemma.

We get the following theorem.

Theorem 6. *Let U be a relatively compact open subset of X, and σ be the positive measure defined in the first paragraph. Then there exists a non-negative integral kernel function $k(x, \theta)$ on $U \times \partial U$ such that*

a) *for every $x \in U$, $k(x, \theta)$ belongs to $L^2(\sigma)$;*

b) *for every $\theta \in \partial U$, $k(x, \theta)$ is harmonic in U;*

c) *a function u belongs to the class $R^1(U)$ if and only if*

$$u(x)=\int_{\partial U} f(\eta)k(x, \eta)d\sigma(\eta)$$

on U for some $f \in L^1(\sigma)$.

Proof. The existing of a non-negative kernel function $k(x, \theta)$ with the first property a) is obvious by proceding Theorem 4. Next consider that $k(x, \theta)$ satisfies the property b). Let us now recall that $\mathfrak{H}(X)$ is complete with respect to the topology of compact convergence if there exists an $\mathfrak{H}$-sweeping system on X. Then according above Lemma 5, we have that there exists a sequence $\{F_n^\theta\}$ of compact sets containing θ, which converges to $\{\theta\}$, and thus we may select a sequence $\{F_n^\theta\}$ so that $\mu_x^U(F_n^\theta)/\sigma(F_n^\theta)$ converges to $d\mu_x^U(\theta)/d\sigma(\theta)$, i.e. $k(x, \theta)$, along the topology of compact convergence. Therefore, since the function $x \longrightarrow \mu_x^U(F_n^\theta)/\sigma(F_n^\theta)$ is harmonic in U for every $\theta \in \partial U$, we obtain that the function $x \longrightarrow k(x, \theta)$ is harmonic in U.

The last property c) follows immediately from the relation b) of Theorem 4 and the definition of $R^1(U)$. This completes the proof.

Example.

We will now consider the following example.

Let R^1 be one-dimentional Euclidean space, and X be an open interval $(0, \pi)$ of R^1. Let us denote by $\mathfrak{H}(X)$ the family of solutions of the following equation:

$$u''-u=0 \qquad \text{on } X,$$

that is,

$$\mathfrak{H}(X):=\{x \longrightarrow \alpha\exp(x)+\beta\exp(-x),\ \alpha,\ \beta \in R^1 \text{ and } x \in X\}.$$

Then an open inteval $U=(a, b)$ of $(0, \pi)$ is a regular set. For U and any point $x \in U$, a harmonic measure follows:

$$\mu_x^U=\frac{\exp(b-x)}{\exp(b-a)+1}\varepsilon_a+\frac{\exp(x-a)}{\exp(b-a)+1}\varepsilon_b,$$

where ε_a (resp. ε_b) is a unit point mass at a (resp. b).

Thus a pair $(X, \mathfrak{H}(X))$ satisfies all conditions of the harmonic space in the sense of the paragraph 3. We now construct a positive measure σ as following: for any Borel subset e of ∂U,

$$\sigma(e):=\int_a^b \mu_x^U(e)dx$$
$$=\frac{\exp(b-a)-1}{\exp(b-a)+1}\chi_e(a)+\frac{\exp(b-a)-1}{\exp(b-a)+1}\chi_e(b),$$

where χ_e is a characteristic function of the set e. And so, according to Theorem 6, the integral kernel function $k(x, \theta)$ is implied:

$$k(x, \theta)=\lim_{n\to\infty}\frac{\mu_x^U(e_n^\theta)}{\sigma(e_n^\theta)},$$

where e_n^θ is any Borel subset of ∂U, which contains an arbitrarily given point θ of ∂U for all n and converges to $\{\theta\}$ as n increasing to ∞, that is,

$$e_n^\theta:=\left[\theta-\frac{1}{n},\ \theta+\frac{1}{n}\right]\cap\partial U.$$

Hence we have

$$k(x, \theta)=\lim_{n\to\infty}\frac{\exp(b-x)\cdot\chi_{e_n^\theta}(a)+\exp(x-a)\cdot\chi_{e_n^\theta}(b)}{\{\exp(b-a)-1\}\chi_{e_n^\theta}(a)+\{\exp(b-a)-1\}\chi_{e_n^\theta}(b)}. \tag{1}$$

Then, we obtain particularly that in the case $\theta=a$

$$k(x, a)=\frac{\exp(b-x)}{\exp(b-a)-1}$$

and in the case $\theta=b$

$$k(x, b)=\frac{\exp(x-a)}{\exp(b-a)-1}.$$

Moreover the function

$$x\longrightarrow\frac{\exp(b-x)\cdot\chi_{e_n^\theta}(a)+\exp(x-a)\cdot\chi_{e_n^\theta}(b)}{\{\exp(b-a)-1\}\chi_{e_n^\theta}(a)+\{\exp(b-a)-1\}\chi_{e_n^\theta}(b)}$$

is harmonic and the sequence in (1) is locally equicontinuous in U. Therefore, we have that the function $k(x, \theta)$ is harmonic in U for each $\theta\in\partial U$ and is a Radon-Nikodym derivative $d\mu_x^U(\theta)/d\sigma(\theta)$ on $\bar{U}=[a, b]$.

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