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Note on the Weber-Voetter method

by

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In connection with the Weber-Voetter method for expanding the eigenpolynomial of a square matrix, A. S. Householder has stated the following important fact without detailed proof in his remarkable work [2]: the method leads to the minimal polynomial for the vector e_1 with respect to a given matrix.*) In this note we shall carry out the proof according to the idea of Householder.

1. Introduction.

Let A be an $n\times n$ matrix and let λ be a variable. Then the purpose of the method of Weber-Voetter is, in short, to construct two particular $n\times n$ matrices $W(\lambda)$ and $Q(\lambda)$ which satisfy

$$(A-\lambda I)W(\lambda)=Q(\lambda), \tag{1.1}$$

where I is identity and the elements of matrices $W(\lambda)$ and $Q(\lambda)$ are polynomials of λ. In particular, matrix $Q(\lambda)$ is of the form which can be expressed as follows:

$$Q(\lambda)=\begin{matrix} -f_1(\lambda) & -f_2(\lambda) & -f_3(\lambda) & \cdots\cdots & -f_n(\lambda) \\ \omega_{21} & 0 & 0 & \cdots\cdots & 0 \\ \omega_{31} & \omega_{32} & 0 & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_{n1} & \omega_{n2} & \omega_{n3} \cdots & \omega_{n,n-1} & 0 \end{matrix} \tag{1.2}$$

where elements ω_{ij} ($j=1,2,\ldots,n-1$; $i>j$) are constants and the $(1,j)$ element $-f_j(\lambda)$ is a polynomial of degree j in λ (cf. [1], [2], [3]).

Hence if the reduction (1.1) is possible, it is easy to show that the determinant of the matrix $W(\lambda)$ is a constant, and that $f_n(\lambda)$ is identical with the eigenpolynomial of A apart from a constant factor.

Taking the form of $Q(\lambda)$ into account, the first column of the original matrix $A-\lambda I$ can be used as that of $Q(\lambda)$. Hence for the purpose of carrying through the reduction (1.1), it will very probably be best if the columns in $Q(\lambda)$ are determined in turn one after another beginning with the second. Accordingly, it will be enough if the matrix $W(\lambda)$ can be formed as a product of matrices $M_i(\lambda)$ ($i=2,3,\ldots,n$) as follows:

$$W(\lambda)=M_2(\lambda)M_3(\lambda)\cdots\cdots M_n(\lambda),$$

where $\quad M_i(\lambda)=(e_1,\ldots,\ e_{i-1},\ m_i+\lambda e_{i-1},\ e_{i+1},\ldots,e_n)$;

*) The author could not see the following original papers:
M. Paul, Zur Kenntnis des Weber-Verfahrens, Tech. Hochsch. München, Diplom-Arbeit, 1957.
R. Weber, Sur les méthodes de calcul employées pour la recherche des valeurs et vecteurs propres d'une matrice, Rech. Aéro. 1949, No. 10, 57-60.

here e_i is the i th column of the identity I of order n and m_i is an adequate vector of dimension n. Clearly $M_i(\lambda)$ differs from the identity only in the i th column. Introducing now the following intermediate matrices $P_i(\lambda)$, the reduction (1.1) can be written as

$$P_1(\lambda)=A-\lambda I,$$
$$P_{i+1}(\lambda)=P_i(\lambda)M_{i+1}(\lambda) \quad (i=1,2,\dots,n-1), \quad P_n(\lambda)=Q(\lambda). \tag{1.3}$$

Evidently, in general, such reduction can not be carried through to completion. In connection with this point, we have a general knowledge as follows: in the reduction (1.3), assuming it has been possible to construct $P_m(\lambda)$ $(m<n)$, then the next step can be straight carried out provided

$$e_{m+1}{}^T P_m(\lambda)e_m \neq 0. \tag{1.4}$$

Of course, by our assumption, $e_2{}^T P_m(\lambda)e_1 e_3{}^T P_m(\lambda)e_2 \cdots\cdots e_m{}^T P_m(\lambda)e_{m-1}$ is non-vanishing. Moreover, even if the condition (1.4) does not hold, in the case when there is at least one non-null element $e_k{}^T P_m(\lambda)e_m=\omega_{km}\neq 0$ $(k\geqq m+2)$, $P_m(\lambda)$ can be similarly and effectively transformed by an elementary permutational transformation in order to bring the element ω_{km} to the pivotal position $(m+1,\ m)$. And then, for this transformed matrix, our reduction can be carried out as before. On the contrary, in the unfortunate case when the elements $e_i{}^T P_m(\lambda)e_m$ $(i=m+1,\dots,n;\ m<n)$ all vanish, clearly the effective (non-trivial) reduction finishes at this stage. For such a case, A. S. Householder has stated the following important fact without detailed proof in his remarkable work [2]: the $(1,\ m)$ element of $P_m(\lambda)$ $(m<n)$ is the minimal polynomial for e_1 with respect to the original matrix A.*) In this note we shall carry out the proof according to the idea of Householder.

Certain notational conventions will be kept throughout this note, and these will be listed here. Except for dimensions, indices and functions, lower case Latin letters represent column vectors, lower case Greek letters scalars and capital Latin letters matrices. The superscript T will be used for denoting the transposed vectors and matrices. For a matrix A, $A\begin{pmatrix} i, j, \dots\dots \\ k, l, \dots\dots \end{pmatrix}$ is the minor determinant whose elements are taken from rows $i, j, \dots$ of A and from columns $k,\ l, \dots$ of A, respectively. And also e_i is the i th column of the identity I of order n.

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2. Preliminaries.

We make a start with a simple lemma of factorization as follows:

LEMMA 1. *Let A be an $n\times m$ matrix and let $m\leqq n$. Suppose that every leading principal minor determinant of* $A\begin{pmatrix} 1,2,\dots\dots,m-1 \\ 1,2,\dots\dots,m-1 \end{pmatrix}$ *is non-vanishing. Then A can be expressed in the form*

$$A=LR, \tag{2.1}$$

where L is an $n\times m$ lower trapezoidal matrix and R is an $m\times m$ unit upper triangular matrix; and, when that is so, such a factorization is unique.

PROOF. From our hypothesis, evidently the first $m-1$ column vectors of A are linearly independent. Now, in the case when the m column vectors of A are linearly independent, certainly there exists at least one non-null determinant among $A\begin{pmatrix} 1,2,\dots\dots,m-1,i \\ 1,2,\dots\dots,m-1,m \end{pmatrix}$ $(i=m,$

*) See the preceding footnote.

$m+1,\ldots\ldots,n)$. In this case, without loss of generality, we can assume that $A\begin{pmatrix}1,2,\ldots\ldots,m\\1,2,\ldots\ldots,m\end{pmatrix}\neq 0$. When that is so, it suffices to examine the factorization as follows:

$A=L'DR$, where $L'=(\lambda_{ij})$ is an $n\times m$ unit lower trapezoidal matrix and $D=\text{diag}\ (\delta_1, \delta_2, \ldots\ldots, \delta_m)$ is an $m\times m$ diagonal matrix, and where $R=(\rho_{ij})$ is an $m\times m$ unit upper triangular matrix. It is easily seen from the direct calculations that the problem of the factorization $A=L'DR$ is equivalent to that of solving the following simultaneous system of nm equations in nm variables λ_{ij}, δ_i and ρ_{ij}: $A\begin{pmatrix}1\\1\end{pmatrix}=\delta_1$,

$$A\begin{pmatrix}1,2,\ldots\ldots,k\\1,2,\ldots\ldots,k\end{pmatrix}=A\begin{pmatrix}1,2,\ldots\ldots,k-1\\1,2,\ldots\ldots,k-1\end{pmatrix}\delta_k \qquad (k=2,3,\ldots\ldots,m),$$

$$A\begin{pmatrix}1,2,\ldots\ldots,k-1,k\\1,2,\ldots\ldots,k-1,j\end{pmatrix}=A\begin{pmatrix}1,2,\ldots\ldots,k\\1,2,\ldots\ldots,k\end{pmatrix}\rho_{kj} \qquad (k=1,2,\ldots\ldots,m-1;\ j=k+1,k+2,\ldots\ldots,m),$$

and $$A\begin{pmatrix}1,2,\ldots\ldots,k-1,i\\1,2,\ldots\ldots,k-1,k\end{pmatrix}=A\begin{pmatrix}1,2,\ldots\ldots,k\\1,2,\ldots\ldots,k\end{pmatrix}\lambda_{ik} \qquad (k=1,2,\ldots\ldots,m;\ i=k+1,\ k+2,\ldots\ldots,n).$$

Therefore, it is evident from our hypothesis that there exists a unique solution of the above equations, which implies that the factorization (2.1) is possible and unique.

On the contrary, in the case when the m column vectors of A are linearly dependent, all the determinants $A\begin{pmatrix}1,2,\ldots\ldots,m-1,i\\1,2,\ldots\ldots,m-1,m\end{pmatrix}$ $(i=m,\ m+1,\ \ldots\ldots,\ n)$ vanish. Then it follows instantly from the above equations that $\delta_m=0$ and the elements λ_{im} $(i=m+1,m+2,\ldots,n)$ are all free, and also that the others is uniquely determined. However, even in this case, the m th column of $L=L'D$ vanishes uniquely. Consequently, we see that the pending factorization (2.1) is possible and is uniquely determined. This completes the proof of lemma 1.

Let A be an $n\times n$ matrix and let $e_1,v_2,\ldots\ldots,v_m,\ldots\ldots$ be the Krylov sequence of the initial vector e_1 with respect to A, where $v_{i+1}=A^ie_1$ $(i=1,2,\ldots\ldots)$. For any positive integer j, put $V_j=(e_1, v_2,\ldots\ldots,v_j)$. Clearly V_j is an $n\times j$ rectangular matrix, the columns of which are formed from the first j consecutive vectors in the Krylov sequence described above. Then there is an important lemma which will often be used later.

LEMMA 2. *Let A be an $n\times n$ matrix. For any one positive integer m $(m\leqq n)$, let R_m be an $m\times m$ unit upper triangular matrix and let $Q_{m,0}=(\omega_{ij})$ be an $n\times m$ matrix such that $\omega_{ij}=0$ $(i=2,3,\ldots\ldots,m;\ i\leqq j\leqq m)$ and the others are constants. Consider now the following equation in R_m and $Q_{m,0}$, whose elements are unknown scalars:*

$$AV_mR_m=Q_{m,0}. \tag{2.2}$$

Then, (1) if every leading principal minor determinant of $V_m\begin{pmatrix}1,2,\ldots\ldots,m\\1,2,\ldots\ldots,m\end{pmatrix}$ is non-vanishing, there exists a unique solution of the equation (2.2);

(2) and, when that is so, the following determinantal relations hold:

$$V_{m+1}\begin{pmatrix}1\\1\end{pmatrix}=V_m\begin{pmatrix}1\\1\end{pmatrix}=1,$$

$$V_{m+1}\begin{pmatrix}1,2,\ldots\ldots,t\\1,2,\ldots\ldots,t\end{pmatrix}=V_m\begin{pmatrix}1,2,\ldots\ldots,t\\1,2,\ldots\ldots,t\end{pmatrix}=Q_{m,0}\begin{pmatrix}2,3,\ldots\ldots,t\\1,2,\ldots\ldots,t-1\end{pmatrix}$$
$$=\omega_{21}\omega_{32}\cdots\cdots\omega_{t,t-1}\neq 0 \quad (t=2,3,\ldots\ldots,m), \tag{2.3}$$

and $$V_{m+1}\begin{pmatrix}1,2,\ldots\ldots,m+1\\1,2,\ldots\ldots,m+1\end{pmatrix}=Q_{m,0}\begin{pmatrix}2,3,\ldots\ldots,m+1\\1,2,\ldots\ldots,m\end{pmatrix}=\omega_{21}\omega_{32}\cdots\cdots\omega_{m+1,m} \quad (m<n).$$

PROOF. Let P be an $n\times n$ permutational transformation matrix such that $P=(e_n,e_1,\ldots\ldots,$

e_{n-1}). Then we get

$$PAV_mR_m=PQ_{m,0}. \tag{2.4}$$

Put $L=PQ_{m,0}$, where L is an $n\times m$ lower trapezoidal matrix. Then we see that the problem of lemma 2 is equivalent to that of the factorization of the $n\times m$ matrix PAV_m as follows:

$PAV_m=LR_m^{-1}$, where R_m^{-1} is an $m\times m$ unit upper triangular matrix.

Since this factorization is no other than that of lemma 1, we have only to inquire into the conditions in lemma 1 for the matrix PAV_m.

Clearly $PAV_m=P(v_2,v_3,\ldots\ldots,v_{m+1})$. Hence, the following determinantal relations hold: for each positive integer t $(t=1,2,\ldots\ldots,m-1)$,

$$PAV_m\begin{pmatrix}1,2,\ldots\ldots,t\\1,2,\ldots\ldots,t\end{pmatrix}=V_m\begin{pmatrix}1,2,\ldots\ldots,t+1\\1,2,\ldots\ldots,t+1\end{pmatrix}.$$

Applying now the lemma 1 to the matrix PAV_m, we see that the first part (1) in lemma 2 holds; when that is so, seeing that R_m is a unit upper triangular matrix of order m, it is easily seen from the relation (2.4) that the determinantal relations (2.3) hold.

3. Householder's main theorems.*)

Let $M_i(\lambda)=(e_1,\ldots\ldots,e_{i-1},m_i+\lambda e_{i-1},e_{i+1},\ldots\ldots,e_n)$ $(i=2,3,\ldots\ldots,n)$, where m_i is a vector of dimension n. Then for any positive integer m $(2\leqq m\leqq n)$, the product $M_2(\lambda)M_3(\lambda)\cdots\cdots M_m(\lambda)$ can be expressed as

$$\begin{aligned}&M_2(\lambda)M_3(\lambda)\cdots\cdots M_m(\lambda)=(W_m(\lambda),e_{m+1},\ldots\ldots,e_n),\\&W_m(\lambda)=e_1(1,\lambda,\lambda^2,\ldots\ldots,\lambda^{m-1})+W_1+\lambda W_2+\cdots\cdots+\lambda^{m-2}W_{m-1}\end{aligned} \tag{3.1}$$

where each W_i is an $n\times m$ matrix whose elements are constants and the first i columns of W_i vanish. And then the principal part of the m-stage of reductions (1.1) and (1.3) can be written as follows:

$$(A-\lambda I)W_m(\lambda)=Q_m(\lambda), \tag{3.2}$$

here $Q_m(\lambda)$ is an $n\times m$ matrix and has the form as

$$Q_m(\lambda)=\begin{pmatrix}-f_1(\lambda) & -f_2(\lambda) & -f_3(\lambda)\cdots\cdots & \cdots\cdots & -f_m(\lambda)\\ \omega_{21} & 0 & 0 & \cdots\cdots & 0\\ \omega_{31} & \omega_{32} & 0 & \cdots\cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \omega_{m1} & \omega_{m2} & \omega_{m3}\cdots\cdots & \omega_{m,m-1} & 0\\ \omega_{m+1,1} & \omega_{m+1,2} & \omega_{m+1,3}\cdots\cdots & \omega_{m+1,m-1} & \omega_{m+1,m}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \omega_{n1} & \omega_{n2} & \omega_{n3}\cdots\cdots & \omega_{n,m-1} & \omega_{n,m}\end{pmatrix}, \tag{3.3}$$

where ω_{ij} is constant and $f_j(\lambda)$ is a polynomial of degree j in λ. In this circumstances, we have

THEOREM 1. *Let A be an $n\times n$ matrix. For any one positive integer m $(m\leqq n)$, suppose that every leading principal minor determinant of $V_m\begin{pmatrix}1,2,\ldots\ldots,m\\1,2,\ldots\ldots,m\end{pmatrix}$ is non-vanishing. Then the Weber-Voetter reduction* (3.2) *is possible and two matrices $W_m(\lambda)$ and $Q_m(\lambda)$ in* (3.2) *are uniquely determined by the $m+1$ consecutive Krylov vectors $e_1,v_2,\ldots\ldots,v_{m+1}$.*

*) See the reference [2].

PROOF. In the Weber-Voetter reduction (3.2), $W_m(\lambda)$ can be represented in the form of (3.1) and also $Q_m(\lambda)$ in the form as follows:

$$Q_m(\lambda)=Q_0+\lambda Q_1+\cdots\cdots+\lambda^m Q_m, \tag{3.4}$$

where $Q_0=Q_m(0)$ and Q_i $(i=1,2,\ldots\ldots,m)$ are $n\times m$ matrices such that all the rows of each Q_i except the first vanish. Then it follows from the direct calculations that

$$\begin{aligned}(A-\lambda I)W_m(\lambda)=&A(e_1e_1{}^{*T}+W_1)+\lambda\{A(e_1e_2{}^{*T}+W_2)-(e_1e_1{}^{*T}+W_1)\}\\&+\lambda^2\{A(e_1e_3{}^{*T}+W_3)-(e_1e_2{}^{*T}+W_2)\}+\cdots\cdots\\&+\lambda^{m-2}\{A(e_1e_{m-1}{}^{*T}+W_{m-1})-(e_1e_{m-2}{}^{*T}+W_{m-2})\}\\&+\lambda^{m-1}\{Ae_1e_m{}^{*T}-(e_1e_{m-1}{}^{*T}+W_{m-1})\}-\lambda^m e_1e_m{}^{*T},\end{aligned} \tag{3.5}$$

where $e_i{}^*$ is the i th column of the identity matrix of order m. Accordingly, from the relation (3.2), we get immediately the simultaneous system of linear equations in the elements of W_i $(i=1,2,\ldots\ldots,m-1)$ and Q_i $(i=0,1,2,\ldots\ldots,m)$ as follows:

$$\begin{aligned}&Q_0=A(e_1e_1{}^{*T}+W_1),\\&Q_t=A(e_1e_{t+1}{}^{*T}+W_{t+1})-(e_1e_t{}^{*T}+W_t)\quad(t=1,2,\ldots\ldots,m-2),\\&Q_{m-1}=Ae_1e_m{}^{*T}-(e_1e_{m-1}{}^{*T}+W_{m-1}),\qquad Q_m=-e_1e_m{}^{*T}.\end{aligned} \tag{3.6}$$

To prove the theorem 1, it is enough to show that under the assumptions $V_m\begin{pmatrix}1,2,\ldots\ldots,t\\1,2,\ldots\ldots,t\end{pmatrix}\neq 0$ $(t=1,2,\ldots\ldots,m)$, the system (3.6) has a solution which satisfies the assigned conditions for $W_m(\lambda)$ and $Q_m(\lambda)$ and is uniquely determined by the vectors $e_1,v_2,\ldots\ldots,v_{m+1}$.

From (3.6), it follows at once that $Q_0e_1{}^*=v_2, Q_1e_1{}^*=-e_1$, $Q_te_1{}^*=0$ $(t=2,3,\ldots\ldots,m)$ and $W_te_1{}^*=0$ $(t=1,2,\ldots\ldots,m-1)$. We see hence that the first column of each Q_i and W_i is uniquely determined by the vectors e_1 and v_2. Suppose here that for each t $(t=1,2,\ldots\ldots,k;\ 1\leqq k<m)$, the t th column of each W_i and Q_i was uniquely determined by using the vectors $e_1,v_2,\ldots\ldots,v_{t+1}$. Then it follows directly from (3.6) that

$$\begin{aligned}&Q_0\ e_{k+1}{}^*=AW_1\ e_{k+1}{}^*,\\&Q_t\ e_{k+1}{}^*=AW_{t+1}\ e_{k+1}{}^*-W_t\ e_{k+1}{}^*\quad(t=1,2,\ldots\ldots,k-1),\\&Q_k\ e_{k+1}{}^*=Ae_1-W_k\ e_{k+1}{}^*,\ \ Q_{k+1}e_{k+1}{}^*=-e_1,\\&Q_t\ e_{k+1}{}^*=0\quad(t=k+2,k+3,\ldots\ldots,m).\end{aligned} \tag{3.7}$$

Because of $e_i{}^TQ_t=0$ $(t=1,2,\ldots,m;\ i=2,3,\ldots\ldots,n)$, put $Q_te_{k+1}{}^*=-\xi_{k+1}{}^t e_1$ $(t=1,2,\ldots,k)$. Then it follows from (3.7) that

$$\begin{aligned}&W_t\ e_{k+1}{}^*=AW_{t+1}e_{k+1}{}^*+\xi_{k+1}{}^t e_1\qquad(t=1,2,\ldots\ldots,k-1),\\&W_k\ e_{k+1}{}^*=Ae_1+\xi_{k+1}{}^k e_1,\ \ W_t\ e_{k+1}{}^*=0\quad(t=k+1,k+2,\ldots\ldots,m-1).\end{aligned} \tag{3.8}$$

So that, the first expression of (3.7) and the above relations (3.8) lead to

$$Q_0\ e_{k+1}{}^*=\xi_{k+1}{}^1\ v_2+\xi_{k+1}{}^2\ v_3+\cdots\cdots+\xi_{k+1}{}^k\ v_{k+1}+v_{k+2}. \tag{3.9}$$

From the form of Q_0, $e_i{}^TQ_0e_{k+1}{}^*=0$ $(i=2,3,\ldots\ldots,k+1)$. Hence, we get the following equations in $\xi_{k+1}{}^t$ $(t=1,2,\ldots\ldots,k)$:

$$\sum_{t=1}^{k}\xi_{k+1}{}^t e_i{}^T v_{t+1}+e_i{}^T v_{k+2}=0\quad(i=2,3,\ldots\ldots,k+1). \tag{3.10}$$

By our assumptions, the determinant of the coefficients in (3.10) is different from zero. Hence there exists a solution which is uniquely determined by the vectors $e_1,v_2,\ldots\ldots,v_{k+2}$. Consequently, from the equations (3.7), (3.8) and (3.9), it is easily seen that the $k+1$ st column of each W_i and Q_i is uniquely determined by the vectors $e_1, v_2, \ldots\ldots, v_{k+2}$ and also that of each $W_m(\lambda)$ and $Q_m(\lambda)$ satisfies the assigned conditions. This completes the proof of theorem 1.

For any positive integer m ($m \leqq n$), put

$$K_m(\lambda) = ((A-\lambda I)e_1, (A^2-\lambda^2 I)e_1, \ldots\ldots, (A^m-\lambda^m I)e_1).$$

Then it is easy to show that

$$K_m(\lambda) = (A-\lambda I) V_m (I-\lambda\Gamma_m)^{-1}, \tag{3.11}$$

where Γ_m is an $m \times m$ matrix such that all the superdiagonal elements are equal to one and the others null.

Suppose now that every leading principal minor determinant of $V_m\begin{pmatrix}1,2,\ldots\ldots,m\\1,2,\ldots\ldots,m\end{pmatrix}$ *is non-vanishing.* Then the theorem 1 and the lemma 2 hold: that is, $(A-\lambda I)W_m(\lambda) = Q_m(\lambda)$, $AV_mR_m = Q_{m,0}$, where R_m is an $m\times m$ unit upper triangular matrix and $Q_{m,0}$ is an $n\times m$ matrix, the form of which is alike to that of $Q_m(0)$. In this circumstances, we have

THEOREM 2.
$$\begin{aligned} &Q_m(\lambda) = K_m(\lambda)R_m, \quad Q_m(0) = Q_{m,0}, \\ &W_m(\lambda) = V_m(I-\lambda\Gamma_m)^{-1}R_m. \end{aligned} \tag{3.12}$$

Moreover, for each j ($j=1,2,\ldots,m$), *the* $(1,j)$ *element* $-f_j(\lambda)$ *of* $Q_m(\lambda)$ *can be expressed as follows:*

$$\begin{aligned} &f_1(\lambda) = -\omega_{11} + \lambda, \\ &f_j(\lambda) = -\omega_{1j} + \rho_{1j}\lambda + \cdots\cdots + \rho_{j-1,j}\lambda^{j-1} + \lambda^j \quad (j=2,3,\ldots\ldots,m), \end{aligned} \tag{3.13}$$

where ω_{1j} *is the* $(1,j)$ *element of* $Q_{m,0}$ *and* ρ_{ij} *is the* (i,j) *element of* R_m.

PROOF. Since $K_m(\lambda) = (A-\lambda I)V_m(I-\lambda\Gamma_m)^{-1}$, if it is verified that $K_m(\lambda)R_m$ has the form of $Q_m(\lambda)$ and $V_m(I-\lambda\Gamma_m)^{-1}R_m$ that of $W_m(\lambda)$, it is easily seen from the uniqueness in the theorem 1 that the theorem 2 holds. Now, from the direct calculations, we obtain

$$\begin{aligned} K_m(\lambda)R_m = {} & AV_mR_m + \lambda(AV_m\Gamma_m - V_m)R_m + \lambda^2(AV_m\Gamma_m - V_m)\Gamma_mR_m \\ & + \cdots\cdots + \lambda^{m-1}(AV_m\Gamma_m - V_m)\Gamma_m{}^{m-2}R_m - \lambda^m V_m\Gamma_m{}^{m-1}R_m. \end{aligned}$$

On the other hand, from the lemma 2, we get $AV_mR_m = Q_{m,0}$. Clearly $AV_m\Gamma_m - V_m = -(e_1, 0,\ldots\ldots,0)$. Hence it is easy to see that

$$\begin{aligned} K_m(\lambda)R_m = {} & Q_{m,0} - \lambda(e_1,0,0,\ldots\ldots,0)R_m - \lambda^2(0,e_1,0,\ldots\ldots,0)R_m - \cdots\cdots \\ & - \lambda^m(0,0,0,\ldots\ldots,e_1)R_m. \end{aligned}$$

Put now $R_m = (\rho_{ij})$, where $\rho_{ii}=1$ and $\rho_{ij}=0$ ($i>j$), and also put $Q_{m,0} = (\omega_{ij})$, where $\omega_{ij}=0$ ($i=2,3,\ldots\ldots,m$; $i \leqq j \leqq m$). Then it follows from the above relation that

$$K_m(\lambda)R_m = \begin{pmatrix} -f_1(\lambda) & -f_2(\lambda) & -f_3(\lambda) & \cdots\cdots & -f_m(\lambda) \\ \omega_{21} & 0 & 0 & \cdots\cdots & 0 \\ \omega_{31} & \omega_{32} & 0 & \cdots\cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_{m1} & \omega_{m2} & \omega_{m3} \cdots\cdots & \omega_{m,m-1} & 0 \\ \omega_{m+1,1} & \omega_{m+1,2} & \omega_{m+1,3} \cdots\cdots & \omega_{m+1,m-1} & \omega_{m+1,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega_{n1} & \omega_{n2} & \omega_{n3} \cdots\cdots & \omega_{n,m-1} & \omega_{n,m} \end{pmatrix},$$

where $\quad f_1(\lambda) = -\omega_{11} + \lambda$

and $\quad f_j(\lambda) = -\omega_{1j} + \rho_{1j}\lambda + \cdots\cdots + \rho_{j-1,j}\lambda^{j-1} + \lambda^j \quad (j=2,3,\ldots\ldots,m).$

Further, it follows at once from the direct calculations that the matrix $V_m(I-\lambda\Gamma_m)^{-1}R_m$ is of the same form as that of $W_m(\lambda)$ obtained in the Weber-Voetter reduction, which completes the proof of theorem 2.

4. Minimal polynomial.

THEOREM 3. *Under the same assumption as in the theorem* 1, *the* $m+1$ *consecutive Krylov vectors* $e_1, v_2, \ldots\ldots, v_m$ *and* v_{m+1} *are linearly dependent if and only if, for each* i $(i=m+1, m+2, \ldots\ldots, n)$, $e_i^T P_m(\lambda)e_m=0$, *where* $P_m(\lambda)=(A-\lambda I)M_2(\lambda)\cdots\cdots M_m(\lambda)$. *When that is the case,* $f_m(\lambda)=-e_1^T P_m(\lambda)e_m$ *is the monic minimal polynomial for* e_1 *with respect to* A.

PROOF. To prove sufficiency, suppose first that for all i $(i=m+1, m+2, \ldots\ldots, n)$, $e_i^T P_m(\lambda)e_m=0$. Because of $V_m\begin{pmatrix}1,2,\ldots\ldots,t\\1,2,\ldots\ldots,t\end{pmatrix}\neq 0$ $(t=1,2,\ldots\ldots,m)$, it follows directly from the lemma 2 that there exist two uniquely determined matrices R_m and $Q_{m,0}$ which satisfy the relation

$$A V_m R_m = Q_{m,0}, \tag{4.1}$$

where $R_m=(\rho_{ij})$ is an $m\times m$ unit upper triangular matrix and $Q_{m,0}=(\omega_{ij})$ is an $n\times m$ matrix which $\omega_{ij}=0$ for each pair (i,j) $(i=2,3,\ldots\ldots,m;\ i\leqq j\leqq m)$. Hence we get

$$(A V_m)(R_m e_m^*)=Q_{m,0}e_m^*. \tag{4.2}$$

The left hand side in this expression can be written as

$$\rho_{1m}v_2+\rho_{2m}v_3+\cdots\cdots+\rho_{m-1,m}v_m+v_{m+1}. \tag{4.3}$$

On the other hand, it follows instantly from the theorem 2 that

$$Q_m(0)=Q_{m,0}. \tag{4.4}$$

Evidently our assumption leads to $e_i^T Q_{m,0}e_m^*=e_i^T Q_m(0)e_m^*=e_i^T P_m(\lambda)e_m=0$ $(i=2,3,\ldots\ldots,n)$. So that, the right hand side in the expression (4.2) can be expressed as

$$Q_{m,0}e_m^*=\omega_{1m}e_1. \tag{4.5}$$

Consequently, it is easily seen from (4.2), (4.3) and (4.5) that

$$-\omega_{1m}e_1+\rho_{1m}v_2+\cdots\cdots+\rho_{m-1,m}v_m+v_{m+1}=0 \tag{4.6}$$

But while, from the theorem 2,

$$\begin{aligned} f_m(\lambda) &= -e_1^T Q_m(\lambda)e_m^* = -e_1^T p_m(\lambda)e_m \\ &= -\omega_{1m}+\rho_{1m}\lambda+\cdots\cdots+\rho_{m-1,m}\lambda^{m-1}+\lambda^m. \end{aligned}$$

Hence the identity (4.6) means that $f_m(A)e_1=0$. Because of $V_m\begin{pmatrix}1,2,\ldots\ldots,m\\1,2,\ldots\ldots,m\end{pmatrix}\neq 0$, the m consecutive Krylov vectors $e_1, v_2, \ldots\ldots, v_m$ are linearly independent, which is to say that $f_m(\lambda)$ is the monic minimal polynomial for e_1 with respect to A. This completes the proof of sufficiency.

Conversely, suppose that $e_1, v_2, \ldots\ldots, v_m$ and v_{m+1} are linearly dependent. Because of $V_m\begin{pmatrix}1,2,\ldots\ldots,m\\1,2,\ldots\ldots,m\end{pmatrix}\neq 0$, it is evident that v_{m+1} can be written as a linear combination

$$v_{m+1}=\alpha_1 e_1+\alpha_2 v_2+\cdots\cdots+\alpha_m v_m, \tag{4.7}$$

where α_i $(i=1,2,\ldots\ldots,m)$ are scalars. On the other hand, by applying again the theorem 2 and the lemma 2, we get relations (4.1) and (4.4). Hence also the relation (4.2) holds: that is,

$$-\omega_{1m}e_1+\rho_{1m}v_2+\cdots\cdots+\rho_{m-1,m}v_m+v_{m+1}=\sum_{i=m+1}^{n}\omega_{im}e_i.$$

Taking the relation (4.7) into account, we obtain immediately

$$\delta_1 e_1+\delta_2 v_2+\cdots\cdots+\delta_m v_m=\sum_{i=m+1}^{n}\omega_{im}e_i, \tag{4.8}$$

where $\delta_1=-\omega_{1m}+\alpha_1$ and $\delta_i=\rho_{i-1,m}+\alpha_i$ $(i=2,3,\ldots\ldots,m)$. Let $e_1=(e_1',0)^T$ and $v_i=(v_i',v_i'')^T$ $(i=2,3,\ldots,m)$ be the partitions of e_1 and v_i corresponding to one another, where e_1' and $v_i'$are m dimensional vectors. Then equation (4.8) is equivalent to the system of the following two equations:

$$\begin{aligned}&\delta_1 e_1'+\delta_2 v_2'+\cdots\cdots+\delta_m v_m'=0,\\ &\qquad\delta_2 v_2''+\cdots\cdots+\delta_m v_m''=(\omega_{m+1,m},\ \omega_{m+2,m},\ldots\ldots,\omega_{n,m})^T.\end{aligned} \tag{4.9}$$

But while, from our hypothesis, we get clearly $\det(e_1',v_2',\ldots\ldots,v_m')\neq 0$. Therefore, it follows instantly that $\delta_1=\delta_2=\cdots\cdots=\delta_m=0$. Consequently, we see from (4.9) that $\omega_{im}=0$ $(i=m+1, m+2,\ldots\ldots,n)$, which is to say that the conditions of our theorem are necessary.

COROLLARY. *Under the same hypothesis as in the theorem* 3, *the* $m+1$ *consecutive Krylov vectors* $e_1,v_2,\ldots\ldots,v_m$ *and* v_{m+1} *are linearly independent if and only if there is at least one positive integer* i_m *such that* $m+1\leqq i_m\leqq n$ *and* $e_{i_m}{}^T P_m(\lambda)e_m\neq 0$.

5. Application of the principle of pivot.

In this section, we will strictly apply the principle of pivot to each step of the Weber-Voetter reduction. Then we give attention to the fact that such reduction is not unique. Starting with $A-\lambda I$, put $P_1(\lambda)=A-\lambda I$. And let $\omega_{i_1 1}$ be an element of maximal modulus among the elements $e_i{}^T P_1(\lambda)e_1$ $(i=2,3,\ldots\ldots,n)$ of the first column of $P_1(\lambda)$. If $\omega_{i_1 1}=0$, the effective reduction finishes in this first stage. While, if $\omega_{i_1 1}\neq 0$, let $P_1'(\lambda)$ be the matrix similarly transformed by applying a suitable elementary permutational transformation to $P_1(\lambda)$ in order to bring the element $\omega_{i_1 1}$ to the pivotal position (2,1). For this matrix $P_1'(\lambda)$, the second step of reduction is carried out. The matrix thus obtained can be expressed as $P_1'(\lambda)M_2(\lambda)=P_2(\lambda)$. Let $\omega_{i_2 2}$ be an element of maximal modulus among the elements $e_i{}^T P_2(\lambda)e_2$ $(i=3,4,\ldots\ldots,n)$. If $\omega_{i_2 2}=0$, our reduction finishes in this step. And if $\omega_{i_2 2}\neq 0$, then $P_2(\lambda)$ is similarly transformed by an adequate elementary permutational transformation in order to bring the element $\omega_{i_2 2}$ to the pivotal position (3,2) of the transformed matrix $P_2'(\lambda)$. And so on. Thus, each step of the reduction described above can be written as follows:

$$\begin{aligned}&P_1(\lambda)=A-\lambda I, &&P_1'(\lambda)=I_{2i_1}P_1(\lambda)I_{2i_1},\\ &P_2(\lambda)=P_1'(\lambda)M_2(\lambda), &&P_2'(\lambda)=I_{3i_2}P_2(\lambda)I_{3i_2},\\ &\cdots\cdots\cdots\cdots\cdots\cdots &&\cdots\cdots\cdots\cdots\cdots\cdots\\ &P_t(\lambda)=P_{t-1}'(\lambda)M_t(\lambda), &&P_t'(\lambda)=I_{t+1,i_t}P_t(\lambda)I_{t+1,i_t},\\ &\cdots\cdots\cdots\cdots\cdots\cdots &&\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned} \tag{5.1}$$

where $I_{ij}=I-(e_i-e_j)(e_i-e_j)^T$ $(i,j=1,2,\ldots\ldots,n)$. Moreover it is easily seen that for each t $(t\leqq n)$,

$$\begin{aligned}&e_i{}^T P_t'(\lambda)e_j=e_i{}^T P_t(\lambda)e_j &&(i,j=1,2,\ldots\ldots,t),\\ &|e_i{}^T P_t'(\lambda)e_j|\leqq|e_{j+1}{}^T P_t'(\lambda)e_j| &&(j=1,2,\ldots\ldots,t;\ j+2\leqq i\leqq n).\end{aligned} \tag{5.2}$$

Suppose now that the reduction (5.1) *was effectively carried out up to the m-stage* $(m\leqq n)$ *making a start with the first.*

Put
$$A^{(m)}=T_m^{-1}AT_m, \tag{5.3}$$
and also put
$$M_t^{(m)}(\lambda)=T_m^{-1}T_{t-1}M_t(\lambda)T_{t-1}^{-1}T_m \quad (t=2,3,\ldots\ldots,m), \tag{5.4}$$
where
$$T_k=I_{2i_1}I_{3i_2}\cdots\cdots I_{k+1,i_k} \qquad (k=1,2,\ldots\ldots,m).$$
Clearly $A^{(m)}$ is an $n\times n$ matrix and $M_t^{(m)}(\lambda)$ is alike to $M_t(\lambda)$ except only the permutation of the last $n-t$ elements of the t th column. If we now introduce the intermediate matrices $P_t^{(m)}(\lambda)$ $(t=1,\ 2,\ldots\ldots,m)$, the reduction described above can be expressed as
$$\begin{aligned} &P_1^{(m)}(\lambda)=A^{(m)}-\lambda I, \qquad P_2^{(m)}(\lambda)=P_1^{(m)}(\lambda)M_2^{(m)}(\lambda),\ldots\ldots\ldots, \\ &P_m^{(m)}(\lambda)=P_{m-1}^{(m)}(\lambda)M_m^{(m)}(\lambda)=(A^{(m)}-\lambda I)M_2^{(m)}(\lambda)\cdots\cdots M_m^{(m)}(\lambda), \end{aligned} \tag{5.5}$$
where $P_m'(\lambda)=P_m^{(m)}(\lambda)$ and $P_t^{(m)}(\lambda)$ has the same form as those of matrices $P_t(\lambda)$ and $P_t'(\lambda)$ $(t=1,\ 2,\ldots\ldots,m)$ in the reduction (5.1). Further it is evident from the relations (5.2) that for each t $(t=1,2,\ldots\ldots,m)$,
$$\begin{aligned} &e_i^TP_t^{(m)}(\lambda)e_j=e_i^TP_t'(\lambda)e_j=e_i^TP_t(\lambda)e_j \quad (i,j=1,2,\ldots\ldots,t), \\ &e_i^TP_t^{(m)}(\lambda)e_j=e_i^TP_t'(\lambda)e_j \qquad (i,j=1,2,\ldots\ldots,t+1), \end{aligned} \tag{5.6}$$
and
$$|e_i^TP_t^{(m)}(\lambda)e_j|\leqq|e_{j+1}^TP_t^{(m)}(\lambda)e_j| \quad (j=1,2,\ldots\ldots,t;\ j+2\leqq i\leqq n).$$

Accordingly, our assumption implies that the Weber-Voetter reduction (5.5) *for* $A^{(m)}$ *can be straight and effectively carried through up to the m-stage beginning with the first.* Then it is evident that
$$T_me_1=e_1. \tag{5.7}$$
Let $e_1,v_2^{(m)},\ldots\ldots,v_i^{(m)},\ldots\ldots$ be the sequence of the Krylov vectors for e_1 with respect to $A^{(m)}$. It follows clearly that for each i $(i=2,3,\ldots\ldots)$,
$$v_i^{(m)}=T_m^{-1}v_i. \tag{5.8}$$
Put $V_j=(e_1,v_2,\ldots\ldots,v_j)$ and put $V_j^{(m)}=(e_1,v_2^{(m)},\ldots\ldots,v_j^{(m)})$ for any positive integer j. We get clearly
$$V_j^{(m)}=T_m^{-1}V_j \quad (j=1,2,\ldots\ldots). \tag{5.9}$$
We see therefore that $V_j^{(m)}$ is of rank j if and only if V_j is so. Under the above assumption, we have

LEMMA 3. *The Weber-Voetter reduction* (5.5) *for* $A^{(m)}$ *is straight and effectively carried through up to the m-stage starting off with the first if and only if every leading principal minor determinant of* $V_m^{(m)}\begin{pmatrix}1,2,\ldots\ldots,m\\1,2,\ldots\ldots,m\end{pmatrix}$ *is non-vanishing; and, when that is so, the reduction is unique.*

PROOF. By applying the theorems 1, 2 and the lemma 2 to $A^{(m)}$, it is easily seen that the condition in lemma 3 is sufficient. Hence it may be enough to show the necessity of the condition.

Suppose now that the reduction (5.5) for $A^{(m)}$ was straight and effectively carried out up to the m-stage. Because of $V_m^{(m)}\begin{pmatrix}1\\1\end{pmatrix}=1$, assume that there was a positive integer k such that
$$1\leqq k<m,$$
$$V_m^{(m)}\begin{pmatrix}1,2,\ldots\ldots,t\\1,2,\ldots\ldots,t\end{pmatrix}\neq 0 \quad (t=1,2,\ldots\ldots,k) \text{ and } V_m^{(m)}\begin{pmatrix}1,2,\ldots\ldots,k+1\\1,2,\ldots\ldots,k+1\end{pmatrix}=0.$$
Applying the theorems 1, 2 and the lemma 2 to $A^{(m)}$, we see that the reduction (5.5) for $A^{(m)}$ can be straight, effectively and uniquely carried out up to the k-stage beginning with the first. Obviously the principal part of this reduction can be represented as $(A^{(m)}-\lambda I)W_k^{(m)}(\lambda)=Q_k^{(m)}(\lambda)$, where $W_k^{(m)}(\lambda)$ and $Q_k^{(m)}(\lambda)$ are $n\times k$ matrices corresponding to those of the reduction (3.2) of

A in section 3. On the other hand, applying the lemma 2 to $A^{(m)}$, we see that there exist two uniquely determined matrices $R_k^{(m)}$ and $Q_{k,0}^{(m)}$ which satisfy the relation $A^{(m)}V_k^{(m)}R_k^{(m)}=Q_{k,0}^{(m)}$, where $R_k^{(m)}$ is a $k\times k$ unit upper triangular matrix and $Q_{k,0}^{(m)}=(\omega_{ij}^{(m)})$ is an $n\times k$ matrix having the same property as that of $Q_{k,0}$ in the lemma 2: that is, $\omega_{ij}^{(m)}=0$ $(i=2,3,\ldots\ldots,k;\ i\leqq j\leqq k)$. Applying the theorem 2 to $A^{(m)}$, we get immediately $Q_k^{(m)}(0)=Q_{k,0}^{(m)}$. On account of our assumption and the property (2) of lemma 2 for $A^{(m)}$, it is easily seen that $\omega_{k+1,k}^{(m)}=0$, which implies that $\omega_{ik}^{(m)}=0$ $(i=k+2,\ k+3,\ldots\ldots,n)$, because our present reduction satisfies the relations (5.6) for $t=k$. Consequently, we see that the effective reduction in (5.5) finishes in this k-stage. Because of $k<m$, this contradicts our hypothesis. Therefore, it is necessary that every leading principal minor determinant of $V_m^{(m)}\begin{pmatrix}1,2,\ldots,m\\1,2,\ldots,m\end{pmatrix}$ is non-null. This completes the proof of lemma 3.

THEOREM 4. *Let A be an $n\times n$ matrix. Suppose that the principle of pivot described at the outset in this section is strictly applied at each stage of the Weber-Voetter reduction. Then the following* (1), (2) *and* (3) *hold*:

(1) *The reduction is effectively carried through up to the m-stage ($m\leqq n$) beginning with the first if and only if the m consecutive Krylov vectors $e_1, v_2,\ldots\ldots,v_m$ for the starting vector e_1 with respect to A are linearly independent.*

(2) *In the case when* (1) *holds, the $m+1$ consecutive Krylov vectors $e_1,v_2,\ldots\ldots,v_m$ and v_{m+1} are linearly dependent if and only if for all i $(i=m+1,\ m+2,\ldots\ldots,n)$, the (i,m) element of $P_m'(\lambda)$ vanishes, where $P_m'(\lambda)$ is the matrix obtained at the m-stage of reduction.*

(3) *In the case when* (2) *holds, the* (1, m) *element $-f_m(\lambda)$ of the matrix $P_m'(\lambda)$ except the negative sign is identical with the monic minimal polynomial of e_1 with respect to A and hence, in general, that of A itself. In particular if $m=n$, $f_n(\lambda)$ is clearly the eigenpolynomial of A.*

PROOF. In the (1) of our theorem, it is evident that the linear independency of the m consecutive Krylov vectors $e_1,v_2,\ldots\ldots,v_m$ is necessary. In fact, under our circumstances, it follows directly from the lemma 3 that every leading principal minor determinant of $V_m^{(m)}\begin{pmatrix}1,2,\ldots\ldots,m\\1,2,\ldots\ldots,m\end{pmatrix}$ is non-vanishing. While, from the relations (5.7) and (5.9), we get $T_m e_1=e_1$ and $V_m=T_mV_m^{(m)}$, where T_m is the permutational transformation associated with the applications of the principle of pivot. This means that $e_1,v_2,\ldots\ldots,v_m$ are linearly independent.

In this circumstances, evidently the linear dependency of the $m+1$ consecutive Krylov vectors $e_1,v_2,\ldots\ldots,v_{m+1}$ for e_1 with respect to the original matrix A is equivalent to that of the $m+1$ consecutive Krylov vectors $e_1,v_2^{(m)},\ldots\ldots,v_{m+1}^{(m)}$ for e_1 with respect to $A^{(m)}$. Applying now the theorem 3 to $A^{(m)}$, we see that the $m+1$ consecutive Krylov vectors $e_1,v_2^{(m)},\ldots\ldots,v_m^{(m)}$ and $v_{m+1}^{(m)}$ are linearly dependent if and only if $e_i^TP_m'(\lambda)e_m=0$ for all i $(i=m+1,m+2,\ldots\ldots,n)$, and also, in this case, that $f_m(\lambda)=-e_1^TP_m'(\lambda)e_m$ is the monic minimal polynomial of e_1 with respect to $A^{(m)}$, which is to say that $f_m(\lambda)$ is that of e_1 with respect to the original matrix A, because the relations (5.7) and (5.8) hold: that is, $T_me_1=e_1$ and $T_mv_i^{(m)}=v_i$ for all i $(i=2,\ 3,\ldots\ldots,m+1)$. Hence clearly $f_m(\lambda)$ is, in general, the monic minimal polynomial of A itself. Consequently, we see that (2) and (3) in our theorem hold.

Finally, to prove the sufficiency in the (1) of our theorem, suppose that $e_1,v_2,\ldots\ldots,v_m$ are linearly independent. Then certainly there is a permutational transformation P such that every leading principal minor determinant of $PV_m\begin{pmatrix}1,2,\ldots\ldots,m\\1,2,\ldots\ldots,m\end{pmatrix}$ is non-null. By applying the theorems 1,2 and the lemma 2 to PAP^{-1}, we see that the Weber-Voetter reduction for PAP^{-1} can be carried through up to the m-stage starting off with the first. Hence, judging from the above discussions, it seems that the linear independency of Krylov vectors $e_1, v_2, \ldots\ldots, v_m$ is sufficient for our particular reduction (5.1) described at the outset in this section to be carried out up to the m-stage beginning with the first. But in this place, we shall show it in the following manner.

First, put $P_1(\lambda)=A-\lambda I$, and let $\omega_{i_1 1}$ be an element of maximal modulus among the elements $e_i^T P_1(\lambda)e_1$ $(i=2,3,\ldots\ldots,n)$ of the first column of $P_1(\lambda)$. Because of $V_1\binom{1}{1}=1$, in the case when $\omega_{i_1 1}=0$, it is evident from the theorem 3 that two vectors e_1 and v_2 are linearly dependent, which implies the fact that the maximal number of the linearly independent consecutive Krylov vectors for e_1 with respect to A is equal to one and hence $m=1$. In this case, by putting $P_1'(\lambda)=P_1(\lambda)$, our reduction finishes in this first stage. On the contrary, in the case when $\omega_{i_1 1}\neq 0$, by applying a suitable elementary permutational transformation to $P_1(\lambda)$, the element $\omega_{i_1 1}$ is transformed in the pivotal position (2,1) of the similarly transformed matrix $P_1'(\lambda)$, which can be written as

$$P_1'(\lambda)=I_{2i_1}P_1(\lambda)I_{2i_1}=A^{(1)}-\lambda I,$$

where $$I_{2i_1}=I-(e_2-e_{i_1})(e_2-e_{i_1})^T \text{ and } A^{(1)}=I_{2i_1}AI_{2i_1}.$$

Hence certainly the first stage of our pending reduction can be effectively carried out.

Assume now that there is a positive integer k $(1\leqq k<m)$ such that our pending effective reduction (5.1) is carried out up to the k-stage beginning with the first and finishes at this stage. Then the result of this reduction is represented by means of the expressions obtained from (5.3), (5.4), (5.5) and (5.6) by putting $m=k$. Applying now the lemma 3 to $A^{(k)}$, it is easily seen that every leading principal minor determinant of $V_k^{(k)}\binom{1,2,\ldots\ldots,k}{1,2,\ldots\ldots,k}$ is non-vanishing. And also it follows from the above assumption that for each positive integer i $(i=k+1,k+2,\ldots\ldots,n)$, the $(i,\ k)$ element of $P_k'(\lambda)$ vanishes, where $P_k'(\lambda)$ is the matrix obtained at the k-stage of reduction. In this circumstances, applying the theorem 3 to $A^{(k)}$, we see that the $k+1$ consecutive Krylov vectors $e_1, v_2^{(k)}, \ldots\ldots, v_{k+1}^{(k)}$ for e_1 with respect to $A^{(k)}$ are linearly dependent, which implies the fact that the $k+1$ consecutive Krylov vectors $e_1, v_2, \ldots\ldots, v_{k+1}$ for e_1 with respect to A are also linearly dependent, because $e_1=T_k e_1$ and $v_i=T_k v_i^{(k)}$ $(i=2,3,\ldots\ldots)$. Taking our hypothesis into account, it follows from the above result that $m<k+1$. This contradicts the assumption that $k<m$. Consequently, we see that our pending reduction can be effectively carried through up to the m-stage starting off with the first. This completes the proof of the sufficiency in the (1) of our theorem.

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