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A property of spectral radius of the residual matrix associated with regular splitting in iterative method

by

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Let A be a non-singular real matrix of order n, where the inverse of A is strictly positive. And further, suppose that A permits regular splittings in the sense of Varga. For such a matrix A, it will be proved that for any given regular splitting $A=A_1-A_2$, $A_2\neq 0$, the spectral radius of the corresponding residual matrix $H=A_1^{-1}A_2$ is always a simple eigenvalue, whether H is reducible or not.

1. Introduction.

Consider a decomposition of a real matrix A which satisfies the following conditions: $A=A_1-A_2$, where these matrices are together of order n, A_1 is non-singular and also A_1^{-1} and A_2 are both non-negative. Any such decomposition of A, according to the definition of Varga, is usually called to be a regular splitting of A. In particular, in the important case when the inverse of A is strictly positive, such regular splittings include most of the cases of interest: that is, those in the Gauss iteration, in the Gauss-Seidel iteration, in the method of under relaxation scheme and in their variations. Varga R. S. have shown that any regular splitting of A gives a convergent process if and only if the inverse of A is non-negative. And moreover, in the case when the inverse of A is strictly positive, to such regular splittings, he extended the theorem of Fiedler-Pták [1] in the sharper form.

In this paper, we will show that if the inverse of A is strictly positive, and also if A permits regular splittings, then for any regular splitting $A=A_1-A_2$, $A_2\neq 0$, the spectral radius of the corresponding residual matrix $H=A_1^{-1}A_2$ is always a simple eigenvalue, whether H is reducible or not.

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2. Preliminaries.

We start with the following simple lemma:

LEMMA. *Let $A>0$ and $B\geqq 0$ be two square matrices of order n, and put $E=AB$. Then for the spectral radius $\rho(E)$ of E, the following relation holds: $\rho(E)=0$ if and only if $B=0$.*

If $B=0$, there is nothing to prove. Hence suppose that $\rho(E)=0$. Then E is clearly non-negative reducible. Since A and B are both non-negative and in particular $A>0$, each column of E is equal to zero, otherwise strictly positive. And moreover, each column of B, corresponding to a column of E that is null, is also equal to zero. Therefore, assuming now that $E \geqq 0$ does not vanish, there is a permutational transformation P such that the appropriate partition of the transformed matrix $P^T EP$ will be of the form $\begin{bmatrix} 0 & E_{12} \\ 0 & E_{22} \end{bmatrix}$, where E_{12} and E_{22} are both strictly positive. Now since E_{22} is irreducible, from the theorem of Perron-Frobenius (cf. [2], [6]) on the non-negative irreducible matrices, it follows directly that the spectral radius $\rho(E_{22})$ of E_{22} is a strictly positive eigenvalue. Hence so is also $\rho(E)$. This contradicts our assumption that $\rho(E)$ is equal to zero. Consequently, E must be null. Hence B also vanishes.

DEFINITION. Let A, A_1 and A_2 be three real square matrices of order n, which satisfy the following conditions: $A = A_1 - A_2$, where A_1 is non-singular and also A_1^{-1} and A_2 are both non-negative. From now on, any such decomposition of A, according to the definition of Varga, will be named to be a regular splitting of A.

In this place, we state the following proposition which will be used later:

PROPOSITION 1. The theorem of Varga (cf. [4], [5]).

Let $A = A_1 - A_2$ be any regular splitting of A. And put $H = A_1^{-1}A_2$. Then for the spectral radius $\rho(H)$ of H, the following relation holds: $0 \leqq \rho(H) < 1$ *if and only if* $A^{-1} \geqq 0$.

The proof of this proposition is due to Varga [5]. Suppose that $\rho(H) < 1$. From our assumption, A_1 is non-singular, $A_1^{-1} \geqq 0$ and obviously $H \geqq 0$. Hence A is expressed as follows: $A = A_1(I - H)$. Since $H \geqq 0$ and $\rho(H) < 1$, by applying the well-known theorem concerned with M-matrices, we see immediately that $I - H$ is non-singular and $(I-H)^{-1} \geqq 0$. Consequently, it follows directly that A is non-singular and $A^{-1} \geqq 0$.

Conversely, suppose here that A is non-singular and $A^{-1} \geqq 0$. Then H can be written as follows: $H = (I+E)^{-1}E$, $E = A^{-1}A_2$. Therefore, eigenvalues of H and of E, when properly paired, are related by $\lambda(H) = \dfrac{\lambda(E)}{1+\lambda(E)}$. Now $f(\lambda) = \dfrac{\lambda}{1+\lambda}$ is a strictly monotonically increasing function of λ in the interval $[0, \infty)$ and also the range of $f(\lambda)$ is an interval $[0, 1)$. Since matrices H and E are both non-negative, by applying the extended result of the Perron-Frobenius theorem for non-negative irreducible matrices, it follows that these spectral radii $\rho(H)$ and $\rho(E)$ are eigenvalues of H and E, respectively. Hence, from the monotonicity of $f(\lambda)$, we get immediately that $\rho(H) = \dfrac{\rho(E)}{1+\rho(E)}$ and $0 \leqq \rho(H) < 1$. This completes the proof.

COROLLARY. *In proposition* 1, *if* $A^{-1} > 0$, *then* $\rho(H) = 0$ *if and only if* $A_2 = 0$.

In fact, from the proof of proposition 1, it is easy to see that $\rho(H) = 0$ is equivalent to $\rho(E) = 0$. On the other hand, seeing that $A^{-1} > 0$, it follows directly from the lemma that $\rho(E) = 0$ implies $A_2 = 0$.

3. Simplicity of spectral radius as eigenvalue.

PROPOSITION 2. *Let A be a non-singular real matrix of order n, satisfying the conditions*: $A^{-1} > 0$ *and A permits regular splittings. Then, for any regular splitting $A = A_1 - A_2$, if $A_2 \neq 0$, the spectral radius $\rho(H)$ of $H = A_1^{-1}A_2$ is a simple eigenvalue of H.*

By our assumption, $A^{-1} > 0$ and also A_2 is non-negative and non-null. Hence H can be written as follows:

$$H=(I+E)^{-1}E, \qquad E=A^{-1}A_2,$$

where E is non-negative and does not vanish. While, from the proposition 1 and the corollary, it follows directly that $0<\rho(H)=\gamma<1$. Moreover, applying the extended result of the Perron-Frobenius theorem concerned with non-negative irreducible matrices to H, there is at least one non-negative and non-null vector c such that

$$Hc=\gamma c, \qquad \gamma=\rho(H).$$

Then it is easy to see that

$$Ec=\frac{\gamma}{1-\gamma}c.$$

First of all, we will prove that c is strictly positive, whether E is reducible or not.

Now, let E be irreducible. Then, if c has at least one null element, by applying the suitable permutational transformation P if necessary, c is clearly partitioned as follows: $Pc=\begin{bmatrix} c_1 \\ 0 \end{bmatrix}$, $c_1>0$. Let $\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$ be the corresponding partition of PEP^T. Then it follows instantly that $E_{21}c_1=0$. Since $c_1>0$ and $E_{21}\geqq 0$, E_{21} must be null. This contradicts that E is irreducible. Consequently, we see that c must be strictly positive.

Next, suppose that E is reducible. Seeing that $E=A^{-1}A_2$ is non-negative and $A^{-1}>0$, and moreover, applying an adequate permutational transformation P to E if necessary, the same argument in the lemma of section 2 leads us to the following partition: $P^TEP=\begin{bmatrix} 0 & E_{12} \\ 0 & E_{22} \end{bmatrix}$, where E_{12} and E_{22} are both strictly positive. Put $d=P^Tc$. Then d is non-negative and non-null. Let $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ be the partition of d, corresponding to that of P^TEP. And then the following relations hold:

$$(1) \quad E_{12}d_2=\frac{\gamma}{1-\gamma}d_1, \qquad (2) \quad E_{22}d_2=\frac{\gamma}{1-\gamma}d_2.$$

Since E_{22} is positive, if d_2 has at least one null element, then d_2 must be null.

In fact, assuming that d_2 has non-null elements, there is a permutational transformation Q such that the first i elements of Qd_2 are strictly positive and the others null. Let $Qd_2=\begin{bmatrix} f \\ 0 \end{bmatrix}$, $f>0$, be the partition of Qd_2 and let $QE_{22}Q^T=\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$ be the corresponding partition of $QE_{22}Q^T$. Then it follows directly that $F_{21}f=0$. This contradicts the fact that $F_{21}>0$ and $f>0$. Hence d_2 must be null.

Consequently, the relation (1) shows that d_1 must be null. And then evidently d also vanishes. But while, by our assumption, d is non-null. This contradiction implies that d_2 is strictly positive. So that, on account of the relation (1), d_1 is also strictly positive and then so is also d. Hence, it is obviously true that c is strictly positive and, seeing the relation (2), $\rho(E)=\rho(E_{22})=\frac{\gamma}{1-\gamma}$.

Let $R_+=\{c\}$ be the set of all positive eigenvectors of H. From the fact that was mentioned above, R_+ is not empty. And moreover, there exists a positive vector c such that $Hc=\gamma c$, where $\gamma=\rho(H)>0$. Then it can be proved that any one vector belonging to R_+ is expressed as an adequate positive scalar multiple of c, and further matrix H belongs to the class (M) of matrices in the sense of Householder [3: Chap. 2].

Assuming now that c' is any one vector belonging to R_+, there is an eigenvalue β of

H, satisfying the relation $Hc'=\beta c'$. Then β is a non-negative real number. Suppose that $\beta=0$, then Ec' must be null. But E is non-negative and c' is strictly positive. Therefore E must be null. This contradicts that E is non-null. Consequently β must be a positive number.

Here, considering c'-norm of matrix H, it is easy to see that $\|H\|_{c'}=\beta$. On the other hand, applying the well-known theorem on the norm of matrices, it is evident that

$$\gamma=\rho(H)\leqq\|H\|_{c'}.$$

Hence, on account of the fact that, by the definition of $\rho(H)$, $\beta\leqq\rho(H)$, it follows immediately that

$$\beta=\gamma,\ \ Hc'=\gamma c'\ \text{ and }\ \|H\|_{c'}=\rho(H)=\gamma.$$

Therefore, certainly H belongs to class (M) of matrices.

Now, if $c'=c$, there is nothing to prove. Hence, suppose in this place that c' is not equal to c. Then there exists the infimum α_0 of the set of all real number α which satisfies the inequality: $\alpha c\geqq c'$. Put $b=\alpha_0 c-c'$. Then b is clearly non-negative and has at least one null element. And further, it satisfies the following equation: $Hb=\gamma b$. Consequently, applying again the same argument already described above, we see that b must be null. And then $c'=\alpha_0 c$, $\alpha_0>0$. This completes the proof for our present question.

From the above results concerning the set R_+, it is easy to see that the rank of matrix $\gamma I-H$ is equal to $n-1$. Hence, in the complex vector space C^n of dimension n, the complex dimension of the eigenspace, belonging to the eigenvalue γ, is also equal to one. On the other hand, since H belongs to class (M) of matrices, the well-known theorem on matrices of class (M) asserts that the complex dimension of the eigenspace corresponding to γ is equal to the algebraic multiplicity of γ in the eigenequation of H. Therefore it follows that γ is a simple eigenvalue of H. This completes the proof of proposition 2.

4. Results.

From the facts that was mentioned above, we get the following theorem:

THEOREM. *Let A be a non-singular real matrix of order n which satisfies the conditions: $A^{-1}>0$ and also A permits regular splittings. And, for any regular splitting $A=A_1-A_2$, put $H=A_1^{-1}A_2$. Then, if $A_2\neq 0$, the spectral radius $\rho(H)$ of H is a simple eigenvalue, whether H is reducible or not, and satisfies the following inequality:* $0<\rho(H)<1$.

COROLLARY 1. *Under the same assumptions as the theorem, $\rho(H)=0$ implies* $A_2=0$.

COROLLARY 2. *Let $A=I-B$ be a matrix of order n, where I is identity, B is non-negative irreducible and further the spectral radius of B is smaller than one. Then for any regular splitting of A, the same results as the theorem obviously hold.*

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